Take home resit exam Multivariable analysis 9-4-2020

Always motivate your answers. You can freely use the results from the lecture notes. Please hand in your solutions in latex, scans of handwritten solutions will not be accepted. In addition the board of examiners has asked me to have you print, read, sign and scan the following declaration. Please send the signed declaration and your solutions to our email address `multivariableanalysis19@gmail.com` before the deadline 23-4, 11pm. If some question is unclear or you believe there might be a typo, do not hesitate to contact us.
Good luck!

Declaration of the Board of Examiners

The Board of Examiners has allowed the conversion of your exam into a take-home exam. This conversion comes with additional provisions.

Here are the provisions that are relevant to you sitting the exam:
1. You are required to sign the attached pledge, swearing that your work has been completed autonomously and using only the tools and aids that the examiner has allowed you to use.
2. Attempts at cheating, fraud or plagiarism will be seen as attempts to take advantage of the Corona crisis and will be dealt with very harshly by the board of examiners.
3. The board of examiners grants your examiner the right to conduct a random sampling. If you are selected for this sample, you may be required to conduct a discussion (digitally, using audio and video) in which you are asked to explain and/or rephrase (some of) the answers you submitted for the take-home exam.

I, ...

(enter your name and student number here)

have completed this exam myself and without help from others unless expressly allowed by my lecturer. I have come up with these answers myself. I understand that my fellow students and my lecturers are all doing their best to do their work as well as possible under the unusual circumstances of the Corona pandemic, and that any attempt by myself or my fellow students to use these circumstances to get away with cheating would be undermining those efforts and the necessary trust that this moment calls for.

Signature:

1. Differentiation.

   (a) Show that $f : \mathbb{R}^n \to \mathbb{R}$ given by $f(v) = |v|^2$ is differentiable at every point $p \in \mathbb{R}^n$ directly using the definition 2.2.1 from the lecture notes.

   (b) Compute the derivative at point $p = (a, b)$ of the function $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $f(x, y) = (xy^2 + e^{x+y}, 2y)$ only using the properties of theorem 2.2.1. You are also allowed to use the fact that $\frac{d}{dt}e^t = e^t$.

   (c) Give an example of a function f of two variables whose partial derivatives exist but f is not differentiable in the sense of definition 2.2.1.

2. Prove the following version of the Fubini theorem: For a continuous function $f : R \times S \to \mathbb{R}$ defined on a product of rectangles $R \subset \mathbb{R}^k$ and $S \subset \mathbb{R}^\ell$ the function $G : S \to \mathbb{R}$ is defined by $G(q) = \int_R f(\cdot, q)$ is continuous and we have $\int_{R\times S} f = \int_S G$.

3. Imagine a finite dimensional vector space V and a linear map $A \in L(V, V)$. Prove that if A satisfies $|Av - v| \leq \frac{1}{2}|v|$ for all $v \in V$ then A must be invertible.

4. For $C \subset \mathbb{R}^n$ we say a contraction is a map $\Phi : C \to C$ with the property that there is an $\alpha \in [0, 1)$ such that for all $x, y \in C$ we have $|\Phi(x) - \Phi(y)| < \alpha|x - y|$. Prove that the restriction of a linear map $A \in L(\mathbb{R}^n, \mathbb{R}^n)$ to $C = \{x \in \mathbb{R}^n : |x| \leq 1\}$ is a contraction if and only if $|A| < 1$.

5. Suppose $F : \mathbb{R}^n \to \mathbb{R}^n$ is such that $F'(0)$ is surjective and $F(0) = 0$. Prove that there exists some $r > 0$ such that the open ball $B_r(0)$ is contained in the image of F.

6. Consider the system of 4 equations in 10 unknowns:

$$\begin{aligned} a^2 + b^2 + c^2 + d^2 + e^2 + f^2 + g^2 + h^2 + i^2 + j^2 &= 10 \\ a + b + c + 2d + e &= 6 \\ (1 + ij)\cos(\pi a) + e + c + 2b &= 2 \\ a + b + abchg + ef &= 4 \end{aligned}$$

   Prove that there exists an open ball around the point corresponding to the solution $a = b = c = d = e = f = g = h = i = j = 1$ where set of solutions coincides with the graph of a differentiable function.

7. Imagine a finite dimensional vector space V of dimension at least 2 and non-zero vector $z \in V$.

   (a) Prove that there does not exist a linear map $F : \Lambda^2 V \to \Lambda^2 V$ such that for all $v, w \in V$ we have $F(v \wedge w) = z \wedge w$.

   (b) Prove that $z + 1 \neq 0$ in ΛV, where 1 is the empty wedge product that spans $\Lambda^0 V$.

   (c) Prove that the dimension of the exterior algebra ΛV is $2^{\dim V}$.

8. Consider a non-empty open subset $P \subset \mathbb{R}^n$ and $k \in \mathbb{N}$. Why is the exterior derivative d NOT a map from $\Omega^k(P)$ to $\Omega^{k+1}(P)$?

9. Define $f : \mathbb{R}^3 \to \mathbb{R}^4$ by $f(x, y, z) = (2x + 3y, xz, yz, xy)$ and $\omega, \eta \in \Lambda^2(\mathbb{R}^4)$ by $\omega(a, b, c, d) = ae^1 \wedge e^4 + ce^2 \wedge e^3$ and $\eta(a, b, c, d) = da \wedge dc$.

   (a) Show that $f^*(\omega \wedge \eta) = 0$.

   (b) Express $f^*\omega$ in terms of dx, dy and dz.

   (c) Compute the integrals $\int_{f \circ \gamma} \omega$ and $\int_\gamma f^*\omega$, where $\gamma : [0, 1]^2 \to \mathbb{R}^3$ is defined by $\gamma(s, t) = (s, t, st)$.

   (d) Do you see a relation between the two answers from the previous part?

   (e) Apply Stokes theorem to write $\int_{f \circ \gamma} \eta$ as an integral over a 1-chain.

10. Consider the 3-cube $\gamma : [0, 1]^3 \to \mathbb{R}^4$ defined by

$$\gamma(a, b, c) = (\cos(\pi a), \sin(\pi a) \cos(\pi b), \sin(\pi a) \sin(\pi b) \cos(2\pi c), \sin(\pi a) \sin(\pi b) \sin(2\pi c))$$

   (a) Compute the boundary $\partial\gamma$.

   (b) Find an element $\omega \in \Omega^3(\mathbb{R}^4 - \{0\})$ such that $\int_\gamma \omega \neq 0$

   (c) Referring to part b), can there exist an η such that $d\eta = \omega$?

   (d) Can you choose your ω from part b) to satisfy $d\omega = 0$?

11. (a) Consider the 2-form ω on $\mathbb{R}^3$ defined by $\omega = f dx \wedge dy$ for some C^2-function $f : \mathbb{R}^3 \to \mathbb{R}$. Prove that if $\frac{\partial f}{\partial z} = 0$ then $f dx \wedge dy$ can be written as $d\alpha$ for some α.

   (b) Find two distinct 2-cubes $\alpha, \beta : [0, 1]^2 \to \mathbb{R}^4$ whose boundary is a parametrization of the circle $S = \{(x, y, 0, 0) \in \mathbb{R}^4 | x^2 + y^2 = 1\}$.

   (c) Show that if $\omega \in \Omega(\mathbb{R}^4)$ satisfies $d\omega = 0$ then $\int_{\alpha - \beta} \omega = 0$, where α and β are as in part b).